

# Exact polynomial solutions of second order differential equations and their applications

Yao-Zhong Zhang

*School of Mathematics and Physics, The University of Queensland,  
Brisbane, Qld 4072, Australia*

## Abstract

We find all polynomials  $Z(z)$  such that the differential equation

$$\left\{ X(z) \frac{d^2}{dz^2} + Y(z) \frac{d}{dz} + Z(z) \right\} S(z) = 0,$$

where  $X(z), Y(z), Z(z)$  are polynomials of degree at most 4, 3, 2 respectively, has polynomial solutions  $S(z) = \prod_{i=1}^n (z - z_i)$  of degree  $n$  with distinct roots  $z_i$ . We derive a set of  $n$  algebraic equations which determine these roots. We also find all polynomials  $Z(z)$  which give polynomial solutions to the differential equation when the coefficients of  $X(z)$  and  $Y(z)$  are algebraically dependent. As applications to our general results, we obtain the exact (closed-form) solutions of the Schrödinger type differential equations describing: 1) Two Coulombically repelling electrons on a sphere; 2) Schrödinger equation from kink stability analysis of  $\phi^6$ -type field theory; 3) Static perturbations for the non-extremal Reissner-Nordström solution; 4) Planar Dirac electron in Coulomb and magnetic fields; and 5)  $O(N)$  invariant dectic anharmonic oscillator.

*2000 Mathematics Subject Classification.* 34A05, 30C15, 81U15, 81Q05, 82B23.

*PACS numbers:* 02.30.Hq, 03.65.-w, 03.65.Fd, 03.65.Ge, 02.30.Ik.

*Keywords:* Polynomial solutions, Bethe ansatz, Quasi-exactly solvable systems.

## 1 Introduction and main results

Consider the general 2nd order linear ordinary differential equation (ODE)

$$\left\{ X(z) \frac{d^2}{dz^2} + Y(z) \frac{d}{dz} + Z(z) \right\} S(z) = 0, \quad (1.1)$$

where  $X(z), Y(z), Z(z)$  are polynomials of degree at most 4, 3, 2 respectively,

$$X(z) = \sum_{k=0}^4 a_k z^k, \quad Y(z) = \sum_{k=0}^3 b_k z^k, \quad Z(z) = \sum_{k=0}^2 c_k z^k.$$

The ODE (1.1) has as many as 12 parameters  $a_k, b_k, c_k$ , and contains, as particular cases, the Heun and generalized Heun equations as well as various confluent equations. As examples, here we list five ODEs for each of which we provide a physical application in section 3.

- (i)  $a_4 = b_3 = c_2 = 0$ , i.e.  $\deg X(z) = 3$ ,  $\deg Y(z) \leq 2$  and  $\deg Z(z) \leq 1$ . If  $X(z)$  has no multiple roots, we can write  $X(z) = \prod_{s=1}^3 (z - d_s)$ ,  $\frac{Y(z)}{X(z)} = \sum_{s=1}^3 \frac{\alpha_s}{z - d_s}$  for suitable complex numbers  $d_s, \alpha_s$ . We then obtain the Heun equation,

$$\left\{ \frac{d^2}{dz^2} + \sum_{s=1}^3 \frac{\alpha_s}{z - d_s} \frac{d}{dz} + \frac{Z(z)}{\prod_{s=1}^3 (z - d_s)} \right\} S(z) = 0. \quad (1.2)$$

- (ii)  $\deg X(z) = 4$ ,  $\deg Y(z) \leq 3$  and  $\deg Z(z) \leq 2$ . If  $X(z)$  has no multiple roots, we can write  $X(z) = \prod_{s=1}^4 (z - e_s)$ ,  $\frac{Y(z)}{X(z)} = \sum_{s=1}^4 \frac{\mu_s}{z - e_s}$  for suitable c-numbers  $e_s, \mu_s$ . Then (1.1) takes the form,

$$\left\{ \frac{d^2}{dz^2} + \sum_{s=1}^4 \frac{\mu_s}{z - e_s} \frac{d}{dz} + \frac{Z(z)}{\prod_{s=1}^4 (z - e_s)} \right\} S(z) = 0. \quad (1.3)$$

This is the generalized Heun equation.

- (iii)  $\deg X(z) = 3$  (i.e.  $a_4 = 0$ ),  $\deg Y(z) \leq 3$  and  $\deg Z(z) \leq 2$ . If  $X(z)$  has no multiple roots, we can write  $X(z) = \prod_{s=1}^3 (z - f_s)$ ,  $\frac{Y(z)}{X(z)} = \sum_{s=1}^3 \frac{\nu_s}{z - f_s} + \nu$  for suitable c-numbers  $f_s, \nu_s, \nu$ . Then (1.1) has the form,

$$\left\{ \frac{d^2}{dz^2} + \left( \sum_{s=1}^3 \frac{\nu_s}{z - f_s} + \nu \right) \frac{d}{dz} + \frac{Z(z)}{\prod_{s=1}^3 (z - f_s)} \right\} S(z) = 0. \quad (1.4)$$

This is the equation considered by Schäfke and Schmidt [1].

- (iv)  $\deg X(z) = 2$  (i.e.  $a_4 = a_3 = 0$ ),  $\deg Y(z) \leq 3$  and  $\deg Z(z) \leq 2$ . If  $X(z)$  has no multiple roots, we can write  $X(z) = (z - g_1)(z - g_2)$ ,  $\frac{Y(z)}{X(z)} = \frac{\sigma_1}{z - g_1} + \frac{\sigma_2}{z - g_2} + \sigma z + \kappa$  for suitable c-numbers  $g_1, g_2, \sigma_1, \sigma_2, \sigma, \kappa$ . Then (1.1) is given by

$$\left\{ \frac{d^2}{dz^2} + \left( \frac{\sigma_1}{z - g_1} + \frac{\sigma_2}{z - g_2} + \sigma z + \kappa \right) \frac{d}{dz} + \frac{Z(z)}{(z - g_1)(z - g_2)} \right\} S(z) = 0. \quad (1.5)$$

- (v)  $\deg X(z) = 1$  (i.e.  $a_4 = a_3 = a_2 = 0$ ),  $\deg Y(z) \leq 3$  and  $\deg Z(z) \leq 2$ . Write  $\frac{Y(z)}{X(z)} = \frac{\eta}{z - h} + \lambda z^2 + \gamma z + \delta$  for suitable c-numbers  $h, \eta, \lambda, \gamma, \delta$ . Then (1.1) reads

$$\left\{ \frac{d^2}{dz^2} + \left( \frac{\eta}{z - h} + \lambda z^2 + \gamma z + \delta \right) \frac{d}{dz} + \frac{Z(z)}{(z - h)} \right\} S(z) = 0. \quad (1.6)$$

Recently there is a lot of research interest in finding polynomial solutions to second order differential equations of the form (1.1) [2]- [11]. ODEs with polynomial solutions are often called quasi-exactly solvable and have wide-spread applications in physics, chemistry

and engineering (see e.g. [12] and references therein). One of the classical problems about the ODE (1.1), suggested by E. Heine [13], T. Stieltjes [14] and G. Szego [15], is \*

**Problem** *Given a pair of polynomials  $X(z)$ ,  $Y(z)$  and a positive integer  $n$ ,*

- (a) *find all polynomials  $Z(z)$  such that the ODE (1.1) has a polynomial solution  $S(z)$  of degree  $n$ .*
- (b) *find  $S(z)$ .*

In this paper we solve this problem by means of the so-called Functional (or Analytic) Bethe Ansatz method [16–19]. Precisely, we find the explicit values of the coefficients  $c_2, c_1, c_0$  of  $Z(z)$  which give rise to degree  $n$  polynomial solutions  $S(z)$  of (1.1) with roots  $z_1, z_2, \dots, z_n$  of multiplicity one, and we obtain a set of  $n$  algebraic equations (the so-called Bethe ansatz equations) which determine these roots..

For cases (i) and (ii) above with fixed real  $d_s, e_s$  and positive  $\alpha_s, \mu_s$  numbers in (1.2) and (1.3), respectively, there is a classical result, known as Heine-Stieltjes theorem [13]. This theorem says [5] that if the coefficients of  $X(z)$  and  $Y(z)$  are algebraically independent, i.e. do not satisfy any algebraic relations with integer coefficients, then for an arbitrary positive integer  $n$  there are exactly  $\binom{n + \deg X(z) - 2}{n}$  polynomials  $Z(z)$  of degree exactly  $(\deg X(z) - 2)$  such that the ODE has a degree  $n$  polynomial solution  $S(z)$ . However, even for these two cases, no results about the values of the coefficients  $c_2, c_1, c_0$  of  $Z(z)$  seem to be previously known. Furthermore, one may ask

**Question** *If the coefficients of  $X(z)$  and  $Y(z)$  do satisfy some algebraic relations with integer coefficients, i.e. are algebraically dependent, then how many polynomials  $Z(z)$  are there which lead to degree  $n$  polynomial solutions of the ODE (1.1)?*

To my knowledge, this question was not answered by Heine and Stieltjes in their theorem. In this paper we will also provide an answer to this question as a by-product of our general result (theorem 1.1 below).

We now state one of our main results of this paper.

**Theorem 1.1** *Given a pair of polynomials  $X(z)$  and  $Y(z)$ , then the values of the coefficients  $c_2, c_1, c_0$  of polynomial  $Z(z)$  such that the differential equation (1.1) has degree  $n$  polynomial solution*

$$S(z) = \prod_{i=1}^n (z - z_i) \quad (1.7)$$

*with distinct roots  $z_1, z_2, \dots, z_n$  are given by*

$$c_2 = -n(n-1)a_4 - nb_3, \quad (1.8)$$

---

\*Authors in these references only considered the cases corresponding to (1.2) and (1.3), while the general ODE (1.1) here also contains many other cases, e.g. differential equations (1.4)-(1.6).

$$c_1 = -[2(n-1)a_4 + b_3] \sum_{i=1}^n z_i - n(n-1)a_3 - nb_2, \quad (1.9)$$

$$c_0 = -[2(n-1)a_4 + b_3] \sum_{i=1}^n z_i^2 - 2a_4 \sum_{i<j}^n z_i z_j - [2(n-1)a_3 + b_2] \sum_{i=1}^n z_i - n(n-1)a_2 - nb_1, \quad (1.10)$$

where the roots  $z_1, z_2, \dots, z_n$  satisfy the Bethe ansatz equations,

$$\sum_{j \neq i}^n \frac{2}{z_i - z_j} + \frac{b_3 z_i^3 + b_2 z_i^2 + b_1 z_i + b_0}{a_4 z_i^4 + a_3 z_i^3 + a_2 z_i^2 + a_1 z_i + a_0} = 0, \quad i = 1, 2, \dots, n. \quad (1.11)$$

The above equations (1.8)-(1.11) give all polynomials  $Z(z)$  such that the ODE (1.1) has polynomial solution (1.7).

**Remark 1.2** *T. Stieltjes gave in [14] a set of equations satisfied by the roots of  $S(z)$  for fixed real  $\{d_s\}, \{e_s\}$  and positive  $\{\alpha_s\}, \{\mu_s\}$  numbers in (1.2) and (1.3), respectively. Our Bethe ansatz equations (1.11) reduce to those of Stieltjes under these conditions. However, even for these particular cases, the expressions for the coefficients of  $Z(z)$ , (1.8)-(1.10), had not been obtained previously. Our theorem 1.1 determines the explicit form of all polynomials  $Z(z)$  which give polynomial solution to the general ODE (1.1). From the explicit expressions of  $c_2, c_1, c_0$  above, the number of  $Z(z)$  is given by the number of the solution sets of the Bethe ansatz equations (1.11).*

From (1.8)-(1.11), we also have, as special cases of the theorem 1.1,

**Corollary 1.3** (a) *If for an arbitrary integer  $n$  the coefficients  $a_4$  and  $b_3$  in the ODE (1.1) are algebraically dependent,*

$$2(n-1)a_4 + b_3 = 0, \quad (1.12)$$

*then there are  $n+1$  polynomials  $Z(z)$  with coefficients  $c_2 = n(n-1)a_4$ ,  $c_1 = -n[(n-1)a_3 + b_2]$  [8, 9] and*

$$c_0 = -n[(n-1)a_2 + 2b_1] - 2a_4 \sum_{i<j}^n z_i z_j - [2(n-1)a_3 + b_2] \sum_{i=1}^n z_i,$$

*such that (1.1) has degree  $n$  polynomial solution  $S(z)$  (1.7), where the roots  $z_i$  satisfy (1.11) with  $b_3 = -2(n-1)a_4$ .*

(b) *If  $a_4 = b_3 = c_2 = 0$  and if for an arbitrary integer  $n$  the coefficients  $a_3$  and  $b_2$  are algebraically dependent,*

$$2(n-1)a_3 + b_2 = 0, \quad (1.13)$$

*then there is 1 polynomial  $Z(z) = -n[(n-1)a_3 + b_2]z - n(n-1)a_2 - nb_1$  such that the corresponding ODE has degree  $n$  polynomial solution  $S(z)$  (1.7) with the roots  $z_i$  determined by (1.11) with  $a_4 = b_3 = 0$ ,  $b_2 = -2(n-1)a_3$ .*

Other main results of this paper are the exact (closed-form) solutions of five physical systems (see section 3 below), providing interesting examples and applications of the differential equations (1.2)-(1.6), respectively. The existence of such exact solutions demonstrates that these systems are quasi-exactly solvable.

In the appendix we present some explicit formulas corresponding to the differential equations (1.2)-(1.6).

## 2 The Proofs

*Proof of Theorem 1.1.* We prove theorem 1.1 by using the Functional Bethe ansatz method. This method was used by us to obtain exact polynomial solutions of certain higher order ODEs arising from the nonlinear optical and spin-boson systems [18,19]. Let

$$S(z) = \prod_{i=1}^n (z - z_i), \quad (2.1)$$

be a degree  $n$  polynomial with undetermined, distinct roots  $z_1, z_2, \dots, z_n$ . We will be looking for the values of the coefficients  $c_2, c_1, c_0$  of  $Z(z)$  and the roots  $z_i$ ,  $1 \leq i \leq n$ , such that (2.1) is a solution of (1.1) with fixed coefficients of  $X(z)$  and  $Y(z)$ . Substituting  $Z(z)$  into the ODE and dividing on both sides by  $Z(z)$  gives rise to

$$\begin{aligned} -c_0 &= (a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0) \sum_{i=1}^n \frac{1}{z - z_i} \sum_{j \neq i}^n \frac{2}{z_i - z_j} \\ &\quad (b_3 z^3 + b_2 z^2 + b_1 z + b_0) \sum_{i=1}^n \frac{1}{z - z_i} + c_2 z^2 + c_1 z \end{aligned}$$

The left hand side is a constant and the right hand side is meromorphic function with simple poles at  $z = z_i$  and singularity at  $z = \infty$ . The residues of  $-c_0$  at the simple poles  $z = z_i$  are

$$\begin{aligned} \text{Res}(-c_0)_{z=z_i} &= (a_4 z_i^4 + a_3 z_i^3 + a_2 z_i^2 + a_1 z_i + a_0) \sum_{j \neq i}^n \frac{2}{z_i - z_j} \\ &\quad + b_3 z_i^3 + b_2 z_i^2 + b_1 z_i + b_0. \end{aligned} \quad (2.2)$$

It can then be shown that

$$\begin{aligned} -c_0 - \sum_{i=1}^n \frac{\text{Res}(-c_0)_{z=z_i}}{z - z_i} &= \sum_{i=1}^n [a_4(z^3 + z_i^3 + z^2 z_i + z z_i^2) + a_3(z^2 + z_i^2 + z z_i) \\ &\quad + a_2(z + z_i) + a_1] \sum_{j \neq i}^n \frac{2}{z_i - z_j} \\ &\quad + \sum_{i=1}^n [b_3(z^2 + z_i^2 + z z_i) + b_2(z + z_i) + b_1] + c_2 z^2 + c_1 z \end{aligned}$$

$$\begin{aligned}
&= [n(n-1)a_4 + nb_3 + c_2] z^2 \\
&+ \left[ (2(n-1)a_4 + b_3) \sum_{i=1}^n z_i + n(n-1)a_3 + nb_2 + c_1 \right] z \\
&+ (2(n-1)a_4 + b_3) \sum_{i=1}^n z_i^2 + 2a_4 \sum_{i<j}^n z_i z_j \\
&+ (2(n-1)a_3 + b_2) \sum_{i=1}^n z_i + n(n-1)a_2 + nb_1,
\end{aligned} \tag{2.3}$$

where we have used identities,

$$\begin{aligned}
\sum_{i=1}^n \sum_{j \neq i}^n \frac{1}{z_i - z_j} &= 0, & \sum_{i=1}^n \sum_{j \neq i}^n \frac{z_i}{z_i - z_j} &= \frac{1}{2}n(n-1), \\
\sum_{i=1}^n \sum_{j \neq i}^n \frac{z_i^2}{z_i - z_j} &= (n-1) \sum_{i=1}^n z_i, & \sum_{i=1}^n \sum_{j \neq i}^n \frac{z_i^3}{z_i - z_j} &= (n-1) \sum_{i=1}^n z_i^2 + \sum_{i<j}^n z_i z_j.
\end{aligned}$$

That is

$$\begin{aligned}
-c_0 &= [n(n-1)a_4 + nb_3 + c_2] z^2 \\
&+ \left[ (2(n-1)a_4 + b_3) \sum_{i=1}^n z_i + n(n-1)a_3 + nb_2 + c_1 \right] z \\
&+ \sum_{i=1}^n \frac{\text{Res}(-c_0)_{z=z_i}}{z - z_i} \\
&+ (2(n-1)a_4 + b_3) \sum_{i=1}^n z_i^2 + 2a_4 \sum_{i<j}^n z_i z_j \\
&+ (2(n-1)a_3 + b_2) \sum_{i=1}^n z_i + n(n-1)a_2 + nb_1.
\end{aligned} \tag{2.4}$$

The right hand side of (2.4) is a constant if and only if the coefficients of  $z^2$  and  $z$  as well as all the residues at the simple poles are equal to zero, respectively. This gives  $c_2, c_1$  in terms of the fixed coefficients of  $X(z), Y(z)$  and the roots of  $S(z)$ ,

$$\begin{aligned}
n(n-1)a_4 + nb_3 + c_2 &= 0, \\
(2(n-1)a_4 + b_3) \sum_{i=1}^n z_i + n(n-1)a_3 + nb_2 + c_1 &= 0,
\end{aligned}$$

and the  $n$  algebraic equations determining the roots  $z_i$ ,

$$\sum_{j \neq i}^n \frac{2}{z_i - z_j} + \frac{b_3 z_i^3 + b_2 z_i^2 + b_1 z_i + b_0}{a_4 z_i^4 + a_3 z_i^3 + a_2 z_i^2 + a_1 z_i + a_0} = 0, \quad i = 1, 2, \dots, n. \tag{2.5}$$

These equations are nothing but (1.8), (1.9) and (1.11), respectively. It then follows from (2.4) that

$$\begin{aligned} -c_0 &= (2(n-1)a_4 + b_3) \sum_{i=1}^n z_i^2 + 2a_4 \sum_{i<j}^n z_i z_j \\ &\quad + (2(n-1)a_3 + b_2) \sum_{i=1}^n z_i + n(n-1)a_2 + nb_1, \end{aligned}$$

which is (1.10). The proof is thus completed.  $\square$

*Proof of Corollary 1.3.* Part (b) is obvious. We now prove part (a). If the coefficients of  $X(z)$  and  $Y(z)$  are algebraically dependent,  $b_3 = -2(n-1)a_4$ , then the ODE (1.1) has a hidden  $sl(2)$  algebra symmetry if  $c_2 = n(n-1)a_4$ ,  $c_1 = -n[(n-1)a_3 + b_2]$ . To show this, we write the ODE (1.1) with such coefficients  $c_2, c_1$  as the Schrödinger form

$$H S(z) = -c_0 S(z). \quad (2.6)$$

Then it can be shown that  $H$  is an element of the enveloping algebra of Lie-algebra  $sl(2)$ ,

$$\begin{aligned} H &= a_4 J^+ J^+ + a_3 J^+ J^0 + a_2 J^0 J^0 + a_1 J^0 J^- + a_0 J^- J^- \\ &\quad + \left[ \frac{1}{2}(3n-2)a_3 + b_2 \right] J^+ + [(n-1)a_2 + b_1] J^0 \\ &\quad + \left( \frac{n}{2}a_1 + b_0 \right) J^- - \frac{n^2}{4}a_2 + \frac{n}{2}[(n-1)a_2 + b_1], \end{aligned} \quad (2.7)$$

where

$$J^+ = z^2 \frac{d}{dz} - nz, \quad J^0 = z \frac{d}{dz} - \frac{n}{2}, \quad J^- = \frac{d}{dz} \quad (2.8)$$

are differential operator realization of the  $n+1$  dimensional representation of the  $sl(2)$  algebra.  $-c_0$  is the eigenvalue of  $H$  with polynomial eigenfunction  $S(z) = \prod_{i=1}^n (z - z_i)$ , given by (from theorem 1.1)

$$-c_0 = n[(n-1)a_2 + 2b_1] + 2a_4 \sum_{i<j}^n z_i z_j + [2(n-1)a_3 + b_2] \sum_{i=1}^n z_i, \quad (2.9)$$

and the roots  $z_i$  of the eigenfunction are determined by the Bethe ansatz equations,

$$\sum_{j \neq i}^n \frac{2}{z_i - z_j} + \frac{-2(n-1)a_4 z_i^3 + b_2 z_i^2 + b_1 z_i + b_0}{a_4 z_i^4 + a_3 z_i^3 + a_2 z_i^2 + a_1 z_i + a_0} = 0, \quad i = 1, 2, \dots, n. \quad (2.10)$$

It is well known that the solution space of second order differential operators with a  $sl(2)$  algebraization is  $n+1$  dimensional [8, 9]. Applying this to the  $H$  above, we conclude that the above Bethe ansatz equations have  $n+1$  sets of solutions and thus there  $n+1$  eigenvalues  $-c_0$ , i.e.  $n+1$  polynomials  $Z(z)$ .

□

Let us remark that as far as we know the Schrödinger equation (2.6) with  $H$  given by (2.7) had not been solved previously (except for some very special cases), and thus (2.9) and (2.10) above give the first exact solution for the general ODE (2.6).

### 3 Applications

In this section we apply our general results obtained in section 1 to derive exact solutions of five physical systems, providing examples and applications of the differential equations (1.2)-1.6), respectively.

#### 3.1 Two Coulombically repelling electrons on a sphere

Consider a system of two electrons, interacting via a Coulomb potential, but constrained to remain on the surface of a  $D$ -dimensional sphere of radius  $R$  [20]. The Hamiltonian of the system (in atomic units) is

$$H = -\frac{1}{2} (\nabla_1^2 + \nabla_2^2) - \frac{1}{u}, \quad (3.1)$$

where  $u = |\mathbf{r}_1 - \mathbf{r}_2|$  is the inter-electronic distance. The Schrödinger wave function of the system can be separated as a product of spin, angular and inter-electron wave functions. The inter-electron wave function  $\Psi(u)$  satisfies the ODE [20]

$$\left( \frac{u^2}{4R^2} - 1 \right) \frac{d^2\Psi}{du^2} + \left( \frac{\delta u}{4R^2} - \frac{1}{\gamma u} \right) \frac{d\Psi}{du} + \frac{\Psi}{u} = E\Psi, \quad (3.2)$$

where  $\delta$  and  $\gamma$  are parameters related to the dimension  $D$  of the sphere. Introduce dimensionless variable  $z = \frac{u}{2R}$ . Then the above ODE can be written as

$$\left\{ \frac{d^2}{dz^2} + \left( \frac{1/\gamma}{z} + \frac{\frac{1}{2}(\delta - 1/\gamma)}{z+1} + \frac{\frac{1}{2}(\delta - 1/\gamma)}{z-1} \right) \frac{d}{dz} + \frac{-4R^2 E z + 2R}{z(z+1)(z-1)} \right\} \Psi = 0. \quad (3.3)$$

This ODE has the form of the Heun equation (1.2). It follows from our general results in previous sections that this equation has polynomial solutions of degree  $n = 1, 2, \dots$ ,

$$\Psi(z) = \prod_{i=1}^n (z - z_i), \quad (3.4)$$

where  $z_i$  are the roots of the above polynomial to be determined, provided that  $E$  and  $R$  take the values given by

$$E = \frac{n}{4R^2} (n + \delta - 1), \quad (3.5)$$

$$R = -\frac{1}{2} [2(n-1) + \delta] \sum_{i=1}^n z_i, \quad (3.6)$$



where  $z_1, z_2, \dots, z_n$  obey the Bethe ansatz equations

$$\sum_{j \neq i}^n \frac{2}{z_i - z_j} + \frac{1/\gamma}{z_i} + \frac{\frac{1}{2}(\delta - 1/\gamma)}{z_i + 1} + \frac{\frac{1}{2}(\delta - 1/\gamma)}{z_i - 1} = 0, \quad i = 1, 2, \dots, n. \quad (3.7)$$

The energy eigenvalues  $E$  agree with those obtained in [20] by a different method. However, the above general, exact formula for the radius  $R$  had not been given previously.

For  $n = 1$ , we have, from the Bethe ansatz equations,  $z_1 = \pm \frac{1}{\sqrt{\delta\gamma}}$ . Then  $2R = -\delta z_1 = \sqrt{\frac{\delta}{\gamma}}$  (by choosing the negative root  $z_1 = -\frac{1}{\sqrt{\delta\gamma}}$  so that the radius  $R$  is non-negative),  $E = \gamma$  and the wave function is  $\Psi = z + \frac{1}{\sqrt{\delta\gamma}} = \frac{1}{\sqrt{\delta\gamma}}(1 + \gamma u)$ .

For  $n = 2$ , we find

$$\begin{aligned} z_1 &= \frac{1}{2(\delta + 2)} \left( -\sqrt{2(\delta + 2) + \frac{4\delta + 6}{\gamma}} \pm \sqrt{2(\delta + 2) - \frac{2}{\gamma}} \right), \\ z_2 &= \frac{1}{2(\delta + 2)} \left( -\sqrt{2(\delta + 2) + \frac{4\delta + 6}{\gamma}} \mp \sqrt{2(\delta + 2) - \frac{2}{\gamma}} \right) \end{aligned} \quad (3.8)$$

The other root satisfies  $z_1 = -z_2$  which leads to  $R = 0$  and therefore is discarded. The radius  $R$  and the energy  $E$  are

$$\begin{aligned} R &= -\frac{1}{2}(\delta + 2)(z_1 + z_2) = \frac{1}{2}\sqrt{2(\delta + 2) + \frac{4\delta + 6}{\gamma}}, \\ E &= \frac{\gamma(\delta + 1)}{\gamma(\delta + 2) + 2\delta + 3} \end{aligned} \quad (3.9)$$

and the wave function is

$$\Psi = (z - z_1)(z - z_2) = \frac{1}{\gamma(\delta + 2)} \left[ 1 + \gamma u + \frac{\gamma^2(\delta + 2)}{2\gamma(\delta + 2) + 4\delta + 6} u^2 \right]. \quad (3.10)$$

The ODE (3.3) has a hidden  $sl(2)$  algebra symmetry. To see this, we rewrite (3.3) as

$$H\Psi = -2R\Psi, \quad H = z(z^2 - 1)\frac{d^2}{dz^2} + (\delta z^2 - \frac{1}{\gamma})\frac{d}{dz} - 4R^2 E z. \quad (3.11)$$

If  $E$  takes values given by (3.5), i.e.  $4R^2 E = n(n - 1 + \delta)$ , then  $H$  is combination of  $sl(2)$  algebra generators (2.8),

$$H = J^+ J^- - J^0 J^- + \left[ \frac{1}{2}(3n - 2) + \delta \right] J^+ - \left( \frac{1}{\gamma} + \frac{n}{2} \right) J^-. \quad (3.12)$$

This provides an  $sl(2)$  algebraization of the two electron system. This algebraization had not been realized previously. The eigenvalues  $-2R$  of  $H$  with the polynomial eigenfunction  $\Psi = \prod_{i=1}^n (z - z_i)$  are given (3.6) and the roots  $z_i$  are determined by the Bethe ansatz equations (3.7).

### 3.2 Schrödinger equation from the kink stability analysis of the $\phi^6$ -type field theory

Consider the  $\phi^6$ -type field theory in 1+1 dimensions characterized by the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\lambda \phi \partial^\lambda \phi - \frac{\mu^2}{8g^2(1+\epsilon^2)} (g^2 \phi^2 + \epsilon^2) (1 - g^2 \phi^2)^2, \quad (3.13)$$

where  $\epsilon$  is real dimensionless constant and  $\mu$  has the dimension of mass. As is well-known, the field equation of this theory has a kink solution. When one performs stability analysis around the kink solution one arrives at the Schrödinger equation [21, 22],

$$\left\{ -\frac{d^2}{dx^2} + V(x) \right\} \psi(x) = E\psi(x), \quad (3.14)$$

where  $E \geq 0$  and the potential  $V(x)$  is given by

$$V(x) = \mu^2 \frac{8 \sinh^4 \frac{\mu x}{2} - \left(\frac{20}{\epsilon^2} - 4\right) \sinh^2 \frac{\mu x}{2} + 2 \left(\frac{1}{\epsilon^2} + 1\right) \left(\frac{1}{\epsilon^2} - 2\right)}{8 \left(1 + \frac{1}{\epsilon^2} + \sinh^2 \frac{\mu x}{2}\right)^2}. \quad (3.15)$$

We show that the Schrödinger equation can be transformed into the form of (1.3). To this end, we let

$$\psi = \left(1 + \frac{1}{\epsilon^2} + \sinh^2 \frac{\mu x}{2}\right)^{-\frac{3}{2}} y \quad (3.16)$$

Then it can be shown that  $y$  satisfies the ODE,

$$-y'' + 3\mu \frac{\sinh \frac{\mu x}{2} \cosh \frac{\mu x}{2}}{1 + \frac{1}{\epsilon^2} + \sinh^2 \frac{\mu x}{2}} y' + \frac{\frac{3}{2}\mu^2 \left(1 + \frac{1}{\epsilon^2}\right)}{1 + \frac{1}{\epsilon^2} + \sinh^2 \frac{\mu x}{2}} y = \left(E + \frac{5}{4}\mu^2\right) y. \quad (3.17)$$

Make a change of variable,

$$z = \cosh \frac{\mu x}{2} \quad (3.18)$$

Then the above equation becomes

$$\begin{aligned} & \left[ z^4 + \left(\frac{1}{\epsilon^2} - 1\right) z^2 - \frac{1}{\epsilon^2} \right] y'' - \left[ 5z^3 - \left(\frac{1}{\epsilon^2} + 6\right) z \right] y' \\ & + \left[ \left(\frac{4E}{\mu^2} + 5\right) z^2 + \frac{4E}{\epsilon^2 \mu^2} - \frac{1}{\epsilon^2} - 6 \right] y = 0. \end{aligned} \quad (3.19)$$

This ODE can be easily cast into the form of the generalized Heun equation (1.3), noting that  $X(z) = z^4 + \left(\frac{1}{\epsilon^2} - 1\right) z^2 - \frac{1}{\epsilon^2} = (z+1)(z-1)(z+\frac{i}{\epsilon})(z-\frac{i}{\epsilon})$  has no multiple roots. From our general results in previous sections, the ODE (3.19) has polynomial solutions of degree  $n = 1, 2, \dots$ ,

$$y(z) = \prod_{i=1}^n (z - z_i), \quad (3.20)$$

where  $z_i$  are the roots of the above polynomial to be determined, provided that  $E, \epsilon$  satisfy the relations

$$E = \frac{\mu^2}{4}(n-1)(5-n), \quad (3.21)$$

$$\sum_{i=1}^n z_i = 0, \quad (3.22)$$

$$\frac{6(n-1)}{\epsilon^2} = (n-1)(n-6) + [5-2(n-1)] \sum_{i=1}^n z_i^2 - 2 \sum_{i<j}^n z_i z_j, \quad (3.23)$$

and the Bethe ansatz equations,

$$\left[ z_i^4 + \left( \frac{1}{\epsilon^2} - 1 \right) z_i^2 - \frac{1}{\epsilon^2} \right] \sum_{j \neq i}^n \frac{2}{z_i - z_j} = 5z_i^3 - \left( \frac{1}{\epsilon^2} + 6 \right) z_i, \quad i = 1, 2, \dots, n. \quad (3.24)$$

For  $n = 1$ , we have  $E = 0$  and  $z_1 = 0$  so that  $y(z) = z$ . There is no constraint on  $\epsilon$ , and

$$\psi(x) = \frac{\cosh \frac{\mu x}{2}}{\left( 1 + \frac{1}{\epsilon^2} + \sinh^2 \frac{\mu x}{2} \right)^{\frac{3}{2}}} \quad (3.25)$$

which is the ground state of the system with energy eigenvalue  $E = 0$ .

For  $n = 2$ , we have  $E = \frac{3}{4}\mu^2$  and equations,

$$\frac{6}{\epsilon^2} = -4 + 3(z_1^2 + z_2^2) - 2z_1 z_2, \quad z_1 = -z_2, \quad (3.26)$$

$$\left[ z_1^4 + \left( \frac{1}{\epsilon^2} - 1 \right) z_1^2 - \frac{1}{\epsilon^2} \right] \frac{2}{z_1 - z_2} = 5z_1^3 - \left( \frac{1}{\epsilon^2} + 6 \right) z_1, \quad (3.27)$$

$$\left[ z_2^4 + \left( \frac{1}{\epsilon^2} - 1 \right) z_2^2 - \frac{1}{\epsilon^2} \right] \frac{2}{z_2 - z_1} = 5z_2^3 - \left( \frac{1}{\epsilon^2} + 6 \right) z_2, \quad (3.28)$$

which give  $z_1 = -z_2 = \sqrt{2}$  and  $\frac{1}{\epsilon^2} = 2$ . Thus  $y(z) = (z - \sqrt{2})(z + \sqrt{2}) = z^2 - 2$  and

$$\psi(x) = \frac{\cosh^2 \frac{\mu x}{2} - 2}{\left( 1 + \frac{1}{\epsilon^2} + \sinh^2 \frac{\mu x}{2} \right)^{\frac{3}{2}}}. \quad (3.29)$$

This gives the first excited state eigenfunction. The two energy eigenvalues and eigenfunctions above for  $n = 1, 2$  reproduce those obtained in [21] (from a completely different analysis). Here, we have obtained the closed form expressions for all eigenvalues and eigenfunctions of the system. From (3.21), we see that there are only five non-negative energy solutions to the Schrödinger equation with energy eigenvalues  $E \leq \mu^2$ . All other analytic solutions (corresponding to  $n \geq 6$ ) have negative energy eigenvalues. These negative energy solutions correspond to unstable modes.

### 3.3 Static perturbations for the non-extremal Reissner-Nordström solution

Consider the Lagrangian describing 4-dimensional gravity coupled to an abelian gauge field  $A_\mu$  and a real massive scalar  $\phi$  (with mass  $m_s$ ) [23],

$$\begin{aligned}\mathcal{L} &= \sqrt{g} \left[ \frac{R}{16\pi G} - \frac{1}{2}(\partial_\mu \phi)^2 - \frac{f(\phi)}{4} F_{\mu\nu}^2 - V(\phi) \right], \\ V(\phi) &= \frac{1}{2} m_s^2 \phi^2, \quad f(\phi) = \frac{1}{1 + a^2 \phi^2},\end{aligned}\tag{3.30}$$

where  $R$  is the scalar curvature,  $G$  is the Newton constant, and  $a$  is a parameter which has the dimension of length. The interaction between the gauge and scalar fields is non-renormalizable. For static solution with magnetic charge  $g_m$ , defined by  $\int_{S^2} F = 4\pi g_m$ , where  $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$  is the gauge field 2-form, one must have [23]

$$\begin{aligned}ds^2 &= g_{tt} dt^2 + g_{rr} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \\ F &= g_m d\theta \wedge \sin \theta d\varphi, \quad \phi = \phi(r).\end{aligned}\tag{3.31}$$

The relevant equations of motion are

$$\begin{aligned}\square \phi &= \frac{1}{\sqrt{g}} \partial_r \sqrt{g} g^{rr} \partial_r \phi = \frac{\partial V_{\text{eff}}}{\partial \phi}, \quad V_{\text{eff}}(\phi, r) = V(\phi) + \frac{g_m^2}{2r^4} f(\phi), \\ G_{\mu\nu} &= 8\pi G T_{\mu\nu}.\end{aligned}\tag{3.32}$$

By expressing  $g_{tt} = -e^{2A(r)}$  and  $g_{rr} = e^{2B(r)}$ , one obtains (in the unit  $1/\sqrt{8\pi G} = 1$ ) the following system of equations for  $\phi$  and  $B$ ,

$$\begin{aligned}\phi'' + \left( \frac{2}{r} - 2B' + \frac{r}{2} \phi'^2 \right) &= e^{2B} \frac{\partial V_{\text{eff}}}{\partial \phi}, \\ \frac{1}{2} \phi'^2 + e^{2B} V_{\text{eff}} - \frac{2}{r} B' + \frac{1 - e^{2B}}{r^2} &= 0.\end{aligned}\tag{3.33}$$

Linearizing the above equations around  $\phi = 0$  and fitting  $e^{-2B}$  to the Reissner-Nordström form for large  $r$  gives rise to [23]

$$\begin{aligned}\phi'' + \left( \frac{2}{r} - 2B' \right) \phi' &= e^{2B} \frac{\partial^2 V_{\text{eff}}}{\partial \phi^2}(0, r) \phi, \\ B(r) &= -\frac{1}{2} \log \left( 1 - \frac{M}{4\pi r} + \frac{g_m^2}{2r^2} \right), \quad \frac{\partial^2 V_{\text{eff}}}{\partial \phi^2}(0, r) = m_s^2 - \frac{g_m^2 a^2}{r^4}, \\ \phi &\propto 1 + \frac{2(g_m^2 a^2 - m_s^2)}{g_m^2 - 2} (r - 1) \quad \text{for } r \rightarrow 1, \\ \phi &\propto e^{-m_s r} \quad \text{for } r \rightarrow \infty,\end{aligned}\tag{3.34}$$

where  $M$  is the mass of the Reissner-Nordström black hole. In the non-extremal case  $M^2 > 32\pi^2 g_m^2$ , the function  $r^2 - \frac{M}{4\pi} r + \frac{g_m^2}{2}$  has two roots

$$r_\pm = \frac{M}{8\pi} \pm \frac{1}{8\pi} \sqrt{M^2 - 32\pi^2 g_m^2},\tag{3.35}$$

which correspond to the two horizons of the black hole. Noting that  $r_{\pm}$  obey the relations,

$$r_+ r_- = \frac{g_m^2}{2}, \quad r_+ + r_- = \frac{M}{4\pi}, \quad (3.36)$$

then the ODE for  $\phi$  can be brought into the form,

$$\phi'' + p(r)\phi' + q(r)\phi = 0. \quad (3.37)$$

Here

$$\begin{aligned} p(r) &= \frac{1}{r - r_+} + \frac{1}{r - r_-}, \\ q(r) &= -m_s^2 + \frac{2a^2}{r^2} + 2a^2 \left( \frac{1}{r_+} + \frac{1}{r_-} \right) \frac{1}{r} + \frac{q_+}{r - r_+} + \frac{q_-}{r - r_-} \end{aligned} \quad (3.38)$$

with

$$\begin{aligned} q_+ &= \frac{1}{r_+ - r_-} \left( m_s^2 r_+^2 - \frac{4a^2 r_-^2}{g_m^2} + \frac{m_s^2 g_m^2}{2} (1 - r_+ + r_-) \right), \\ q_- &= \frac{1}{r_+ - r_-} \left( m_s^2 r_-^2 + \frac{4a^2 r_+^2}{g_m^2} + \frac{m_s^2 g_m^2}{2} (1 - r_+ + r_-) \right). \end{aligned} \quad (3.39)$$

The Lagrangian has a rigid rescaling symmetry. This scaling freedom can be used to set the horizon location  $r_+ = 1$ . Then black hole mass  $M$  can be determined by requiring that the horizon is at  $r_+ = 1$ . It follows from (3.36) that  $M = 2\pi(g_m^2 + 2)$  and  $r_- = \frac{g_m^2}{2}$ . The corresponding  $q(r)$  and  $p(r)$  reduce to <sup>†</sup>

$$\begin{aligned} p(r) &= \frac{1}{r - 1} + \frac{1}{r - r_-}, \\ q(r) &= -m_s^2 + \frac{2a^2}{r^2} + 2a^2 \left( 1 + \frac{1}{r_-} \right) \frac{1}{r} \\ &\quad + \frac{a^2 g_m^2 - m_s^2 (1 + r_-^2)}{1 - r_-} \frac{1}{r - 1} + \frac{m_s^2 g_m^2 r_+ + \frac{2a^2}{r_-}}{1 - r_-} \frac{1}{r - r_-}. \end{aligned} \quad (3.40)$$

We now find exact solutions to (3.37) with  $q(r)$ ,  $p(r)$  given by (3.40). By means of transformation

$$\phi(r) = r^\mu e^{-m_s r} f(r), \quad \mu = \frac{1}{2} \left( 1 \pm \sqrt{1 - 8a^2} \right), \quad (3.41)$$

then it can be shown that  $f(r)$  satisfies the following ODE

$$f'' + \left( \frac{2\mu}{r} + \frac{1}{r - 1} + \frac{1}{r - r_-} - 2m_s \right) f' + \frac{c_2 r^2 + c_1 r + c_0}{r(r - 1)(r - r_-)} f = 0, \quad (3.42)$$

---

<sup>†</sup>Authors in [24] also considered the differential equation (3.37) but arrived at the different values of  $M, r_-$  as well as the different  $p(r), q(r)$  functions.

where

$$\begin{aligned}
c_2 &= 2a^2(1 + \frac{1}{r_-}) - 2m_s(\mu + 1) + \frac{1}{1 - r_-} \left( g_m^2(a^2 + m_s^2 r_-) - m_s^2(1 + r_-^2) + \frac{2a^2}{r_-} \right), \\
c_1 &= [m_s(2\mu + 1) + \mu] (r_- + 1) - \frac{2a^2(r_- + 1)^2}{r_-} \\
&\quad - \frac{1}{1 - r_-} \left( g_m^2(a^2 + m_s^2) r_- - m_s^2(1 + r_-^2) r_- + \frac{2a^2}{r_-} \right), \\
c_0 &= 2a^2 + 2[a^2 - \mu(m_s + 1)] r_-.
\end{aligned} \tag{3.43}$$

This ODE is of the form (1.4). Applying our general results in previous sections, we can show that (3.42) has polynomial solutions of degree  $n = 0, 1, 2, \dots$ ,

$$f(r) = \prod_{i=1}^n (r - r_i), \quad f \equiv 1 \text{ for } n = 0, \tag{3.44}$$

where  $r_i$  are the roots of the above polynomial to be determined, provided that  $a, m_s, g_m$  obey the following relations:

$$\begin{aligned}
&2a^2(1 + \frac{1}{r_-}) - 2m_s(\mu + 1) \\
&\quad + \frac{1}{1 - r_-} \left( g_m^2(a^2 + m_s^2 r_-) - m_s^2(1 + r_-^2) + \frac{2a^2}{r_-} \right) = 2m_s n,
\end{aligned} \tag{3.45}$$

$$\begin{aligned}
&[m_s(2\mu + 1) + \mu] (r_- + 1) - \frac{2a^2(r_- + 1)^2}{r_-} \\
&\quad - \frac{1}{1 - r_-} \left( g_m^2(a^2 + m_s^2) r_- - m_s^2(1 + r_-^2) r_- + \frac{2a^2}{r_-} \right) \\
&= 2m_s \sum_{i=1}^n r_i - n[n + 2\mu + 1 + 2m_s(r_- + 1)],
\end{aligned} \tag{3.46}$$

$$\begin{aligned}
&2a^2 + 2[a^2 - \mu(m_s + 1)] r_- \\
&= 2m_s \sum_{i=1}^n r_i^2 - 2[n + \mu + m_s(r_- + 1)] \sum_{i=1}^n r_i \\
&\quad + n[(n - 2\mu)(r_- + 1) - 2m_s r_-],
\end{aligned} \tag{3.47}$$

where the roots  $r_i$  are determined by the set of Bethe ansatz equations,

$$\sum_{j \neq i}^n \frac{2}{r_i - r_j} + \frac{2\mu}{r_i} + \frac{1}{r_i - 1} + \frac{1}{r_i - r_-} - 2m_s = 0, \quad i = 1, 2, \dots, n. \tag{3.48}$$

Note that  $f(r) = 1$  is a solution of the ODE (3.42) provided that  $a, m_s, g_m$  satisfy the following relations

$$2a^2(1 + \frac{1}{r_-}) - 2m_s(\mu + 1) + \frac{1}{1 - r_-} \left( g_m^2(a^2 + m_s^2 r_-) - m_s^2(1 + r_-^2) + \frac{2a^2}{r_-} \right) = 0,$$

$$\begin{aligned}
& [m_s(2\mu + 1) + \mu] (r_- + 1) - \frac{2a^2(r_- + 1)^2}{r_-} \\
& - \frac{1}{1 - r_-} \left( g_m^2(a^2 + m_s^2)r_- - m_s^2(1 + r_-^2)r_- + \frac{2a^2}{r_-} \right) = 0, \\
& 2a^2 + 2 [a^2 - \mu(m_s + 1)] r_- = 0.
\end{aligned} \tag{3.49}$$

These relations can be obtained from (3.45)-(3.47) by letting  $n = 0$ . It follows that

$$\phi(r) = r^{\frac{1}{2}(1 \pm \sqrt{1 - 8a^2})} e^{-m_s r}, \tag{3.50}$$

where  $a$  and  $m_s$  are determined by equations (3.49). This gives the first exact solution of the differential equation (3.37) such that the horizon is at  $r_+ = 1$ .

### 3.4 Planar Dirac electron in Coulomb and magnetic fields

Consider the (2+1)-dimensional relativistic system of a Dirac electron (with mass  $m_e$ ) in the presence of an external electromagnetic field  $A_\mu$ . This system was also examined in [25] via a similar Bethe ansatz approach. The covariant Dirac equation (in the unit  $\hbar = c = 1$ ) has the form

$$i\gamma^\mu(\partial_\mu + ieA_\mu)\Psi(t, \mathbf{r}) = m_e\Psi(t, \mathbf{r}), \tag{3.51}$$

where  $m_e$  is the rest mass of the electron,  $-e$  ( $e > 0$ ) is its electric charge and the  $(2 + 1)$  Dirac gamma matrices  $\gamma^\mu$  satisfy the anti-commutation relations  $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$  with  $\eta^{\mu\nu} = \eta_{\mu\nu} = \text{diag}(1, -1, -1)$ . In an external Coulomb and a constant homogeneous magnetic field  $B$ , the vector potential can be written as

$$A_0 = -\frac{Ze}{r}, \quad A_1 = -\frac{By}{2}, \quad A_2 = \frac{Bx}{2}. \tag{3.52}$$

Then the Hamiltonian  $H(\mathbf{r})$  of the system can be expressed as

$$i\partial_0\Psi(t, \mathbf{r}) = H(\mathbf{r})\Psi(t, \mathbf{r}), \quad H(\mathbf{r}) = \gamma^0\gamma^k P_k + eA_0 + \gamma^0 m_e, \tag{3.53}$$

where  $P_k = -i\partial_k + eA_k$ ,  $k = 1, 2$ , is the operator of generalized momentum of the electron. The wave function  $\Psi(t, \mathbf{r})$  is assumed to have the form

$$\Psi(t, \mathbf{r}) = \frac{1}{\sqrt{r}} \exp(-iEt) \psi_l(r, \theta), \tag{3.54}$$

where  $E$  is the energy eigenvalue of Hamiltonian, and

$$\psi_l(r, \theta) = \begin{pmatrix} F(r) e^{il\theta} \\ G(r) e^{i(l+1)\theta} \end{pmatrix}, \tag{3.55}$$

where  $l$  is an integer. Substituting (3.54) and (3.55) into (3.53) and working in the polar coordinates  $(t, r, \theta)$  reduce the problem to a system of coupled differential equations for  $F(r)$  and  $G(r)$ ,

$$\frac{dF}{dr} - \left( \frac{l + \frac{1}{2}}{r} + \frac{eBr}{2} \right) F + \left( E + m_e + \frac{Z\alpha}{r} \right) G = 0, \quad (3.56)$$

$$\frac{dG}{dr} + \left( \frac{l + \frac{1}{2}}{r} + \frac{eBr}{2} \right) G + \left( E - m_e + \frac{Z\alpha}{r} \right) F = 0, \quad (3.57)$$

where  $\alpha = e^2 = 1/137$  is the fine structure constant. Solving  $G(r)$  from the first equation in terms of  $F(r)$  and substituting into the second equation, we obtain the second-order ODE for  $F(r)$ ,

$$F'' + \left( \frac{1}{r} - \frac{1}{r + r_0} \right) F' + \left\{ E^2 - m_e^2 - eB(l + 1) + \frac{2EZ\alpha + (l + \frac{1}{2})/r_0}{r} - \frac{\frac{eB}{2}r_0 + (l + \frac{1}{2})/r_0}{r + r_0} - \frac{(l + \frac{1}{2})^2 - (Z\alpha)^2}{r^2} - \frac{(eB)^2}{4}r^2 \right\} F = 0, \quad (3.58)$$

where  $r_0 = \frac{Z\alpha}{E + m_e}$ . Applying the transformation,

$$F(r) = r^\xi e^{-eB\frac{r^2}{4}} f(r), \quad \xi = \sqrt{(l + \frac{1}{2})^2 - (Z\alpha)^2}, \quad (3.59)$$

we obtain

$$f'' + \left( \frac{2\xi + 1}{r} - \frac{1}{r + r_0} - eBr \right) f' + \frac{c_2 r^2 + c_1 r + c_0}{r(r + r_0)} f = 0, \quad (3.60)$$

where

$$\begin{aligned} c_2 &= E^2 - m_e^2 - eB(\xi + l + \frac{3}{2}), \\ c_1 &= 2EZ\alpha + \left[ E^2 - m_e^2 - eB(\xi + l + \frac{5}{2}) \right] r_0, \\ c_0 &= 2EZ\alpha r_0 + l + \frac{1}{2} - \xi. \end{aligned} \quad (3.61)$$

This ODE is of the form (1.5) and has, from our general results in previous sections, polynomial solutions of degree  $n = 0, 1, 2, \dots$ ,

$$f(r) = \prod_{i=1}^n (r - r_i), \quad f \equiv 1 \text{ for } n = 0, \quad (3.62)$$

where  $r_i$  are the roots of the above polynomial to be determined, provided that  $E, Z, B$  are given by

$$\begin{aligned} E^2 &= m_e^2 + eB \left( n + l + \xi + \frac{3}{2} \right), \\ 2EZ\alpha &= eB \left( r_0 + \sum_{i=1}^n r_i \right), \\ 2EZ\alpha r_0 &= -n(n + 2\xi - 1) + \xi - (l + \frac{1}{2}) + eB \left( \sum_{i=1}^n r_i^2 + r_0 \sum_{i=1}^n r_i \right), \end{aligned} \quad (3.63)$$



and the Bethe ansatz equations

$$\sum_{j \neq i}^n \frac{2}{r_i - r_j} + \frac{2\xi + 1}{r_i} - \frac{1}{r_i + r_0} - eBr_i = 0, \quad i = 1, 2, \dots, n. \quad (3.64)$$

Let us remark that our (3.63)-(3.64) differ from the corresponding equations (22)-(25) of ref. [25]. It would be interesting to establish the relation between the two sets of expressions.

Note that  $f(r) = 1$  is a solution of the ODE (3.60) provided that  $E, Z, B$  obey the following relations:

$$E^2 = m_e^2 + eB \left( l + \xi + \frac{3}{2} \right), \quad (3.65)$$

$$2EZ\alpha = eBr_0,$$

$$2EZ\alpha r_0 = \xi - \left( l + \frac{1}{2} \right). \quad (3.66)$$

These relations can be obtained from (3.63) by setting  $n = 0$ . Solving these relations we obtain

$$\begin{aligned} eB &= -\frac{m_e^2(l + \frac{1}{2} + \xi)}{(l + 1 + \xi)^2}, \\ E &= -\frac{m_e}{2} + \frac{1}{2}\sqrt{m_e^2 + 2eB} = -\frac{m_e}{2(l + 1 + \xi)}, \end{aligned} \quad (3.67)$$

where  $\xi$  is related to the parameter  $Z$  via the expression given in (3.59). It follows that

$$F(r) = r^\xi e^{-eB\frac{r^2}{4}}, \quad G(r) = \frac{\xi - l - \frac{1}{2} + eBr^2}{(E + m_e)r + Z\alpha} r^\xi e^{-eB\frac{r^2}{4}} \quad (3.68)$$

with  $eB$  and  $E$  given by (3.67) in terms of  $\xi$  (i.e.  $Z$ ). As far as we know, (3.68) gives the first exact solution to the planar Dirac electron system. However, this solution does not seem to be squarely integrable.

### 3.5 Schrödinger equation of $O(N)$ invariant decatic anharmonic oscillator

Consider the  $O(N)$  invariant decatic anharmonic oscillator in  $N$  dimensions [26]. The Schrödinger equation is

$$\left( -\frac{1}{2}\nabla^2 + V(\mathbf{x}) \right) \psi(\mathbf{x}) = E\psi(\mathbf{x}) \quad (3.69)$$

with the potential  $V(\mathbf{x})$  defined by

$$V(\mathbf{x}) = \lambda_1 \mathbf{x}^2 + \lambda_2 (\mathbf{x}^2)^2 + \lambda_3 (\mathbf{x}^2)^3 + \lambda_4 (\mathbf{x}^2)^4 + (\mathbf{x}^2)^5, \quad \mathbf{x}^2 = \sum_{i=1}^N x_i^2. \quad (3.70)$$

Here without loss of generality we have normalized the potential such that the coefficient of  $(\mathbf{x}^2)^5$  equals to 1. In  $N$ -dimensional spherical coordinates, the radial wave function  $R(r)$  satisfies

$$\frac{1}{2} \left[ -\frac{d^2}{dr^2} - \frac{N-1}{r} \frac{d}{dr} + \frac{l(l+N-2)}{r^2} + 2(V(r) - E) \right] R(r) = 0. \quad (3.71)$$

Applying the transformation  $R(r) = r^{\frac{1-N}{2}} \psi(r)$  yields

$$-\psi'' + \left[ \frac{\mu(\mu-1)}{r^2} + 2(V(r) - E) \right] \psi = 0, \quad (3.72)$$

where  $\mu = l + \frac{1}{2}(N-1)$ . Applying the transformation

$$\psi = r^\mu e^{-\alpha \frac{r^2}{2} - \beta \frac{r^4}{4} - \gamma \frac{r^6}{6}} \phi, \quad z = r^2, \quad (3.73)$$

where  $\alpha, \beta, \gamma > 0$  are parameters yet to be determined, yields the ODE for  $\phi(z)$

$$\begin{aligned} \phi'' + \left( \frac{l+N/2}{z} - \gamma z^2 - \beta z - \alpha \right) \phi' + \frac{1}{4z} \left[ (2\alpha\beta - \gamma(N+2l+4) - 2\lambda_2) z^2 \right. \\ \left. + (\alpha^2 - \beta(N+2l+2) - 2\lambda_1) z + 2E - \alpha(N+2l) \right] \phi = 0, \end{aligned} \quad (3.74)$$

where  $\alpha, \beta, \gamma$  are given by

$$\gamma = \sqrt{2}, \quad \beta = \frac{\lambda_4}{\sqrt{2}}, \quad \alpha = \frac{1}{\sqrt{2}} \left( \lambda_3 - \frac{\lambda_4^2}{4} \right) \quad (3.75)$$

The ODE (3.74) is of the form (1.6). Applying our general results in previous section, we obtain that this equation has polynomial solutions of degree  $n = 0, 1, 2, \dots$ ,

$$\phi(z) = \prod_{i=1}^n (z - z_i), \quad \phi \equiv 1 \text{ for } n = 0, \quad (3.76)$$

where  $z_i$  are the roots of the above polynomial to be determined, provided that

$$2\alpha\beta - \gamma(N+2l+4) - 2\lambda_2 = 4\gamma n, \quad (3.77)$$

$$\alpha^2 - \beta(N+2l+2) - 2\lambda_1 = 4\gamma \sum_{i=1}^n z_i + 4\beta n, \quad (3.78)$$

$$E = \frac{\alpha}{2}(4n + N + 2l) + 2\beta \sum_{i=1}^n z_i + 2\gamma \sum_{i=1}^n z_i^2, \quad (3.79)$$

and the Bethe ansatz equations,

$$\sum_{j \neq i}^n \frac{2}{z_i - z_j} = \gamma z_i^2 + \beta z_i + \alpha - \frac{l + N/2}{z_i}, \quad i = 1, 2, \dots, n. \quad (3.80)$$

Note that  $\phi = 1$  is a solution to (3.74) with energy

$$E = \frac{1}{2\sqrt{2}} \left( \lambda_3 - \frac{\lambda_4^2}{4} \right) (N + 2l), \quad (3.81)$$

where  $\lambda_3$  and  $\lambda_4$  obey the constraints,

$$\lambda_4 \left( \lambda_3 - \frac{\lambda_4^2}{4} \right) - \sqrt{2}(N + 2l + 4) - 2\lambda_2 = 0, \quad (3.82)$$

$$\frac{1}{2} \left( \lambda_3 - \frac{\lambda_4^2}{4} \right)^2 - \frac{\lambda_4}{\sqrt{2}}(N + 2l + 2) - 2\lambda_1 = 0, \quad (3.83)$$

which can be obtained from (3.77)-(3.79) by setting  $n = 0$ . The real solutions of these constraint equations (so that the energy  $E$  is real) are given by

$$\begin{aligned} \lambda_4 &= -\frac{2\sqrt{2}\lambda_1}{3(N + 2l + 2)} + \left( -\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} \right)^{1/3} \\ &\quad + \left( -\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} \right)^{1/3}, \\ \lambda_3 &= \frac{\lambda_4^2}{4} + \frac{\sqrt{2}(N + 2l + 4 + \sqrt{2}\lambda_2)}{\lambda_4}, \end{aligned} \quad (3.84)$$

where

$$\begin{aligned} p &= -\frac{24\lambda_1^2}{9(N + 2l + 2)^2}, \\ q &= \frac{32\sqrt{2}\lambda_1^3}{27(N + 2l + 2)^3} - \frac{(N + 2l + 4 + \sqrt{2}\lambda_2)^2}{N + 2l + 2}. \end{aligned} \quad (3.85)$$

We thus obtain the first exact ground state of the radial Schrödinger equation (3.71),

$$R(r) = r^l \exp \left[ -\frac{1}{2\sqrt{2}} \left( \lambda_3 - \frac{\lambda_4^2}{4} \right) r^2 - \frac{\lambda_4}{4\sqrt{2}} r^4 - \frac{1}{3\sqrt{2}} r^6 \right] \quad (3.86)$$

with  $\lambda_3$  and  $\lambda_4$  given by (3.84).

For  $n = 1$ , we find that the real root to the Bethe ansatz equation (3.80) is

$$z_1 = \frac{\lambda_4}{6} + \left( -\frac{v}{2} + \sqrt{\left(\frac{v}{2}\right)^2 + \left(\frac{u}{3}\right)^3} \right)^{1/3} + \left( -\frac{v}{2} - \sqrt{\left(\frac{v}{2}\right)^2 + \left(\frac{u}{3}\right)^3} \right)^{1/3}, \quad (3.87)$$

where

$$\begin{aligned} u &= \frac{\lambda_4^2}{24} - \frac{\lambda_3}{2}, \\ v &= \frac{5\lambda_4^3}{432} - \frac{1}{2}\lambda_3 + \frac{1}{2\sqrt{2}}(N + 2l). \end{aligned} \quad (3.88)$$

The parameters  $\lambda_3$  and  $\lambda_4$  are determined from the equations (see (3.77) and (3.78)),

$$\lambda_4 \left( \lambda_3 - \frac{\lambda_4^2}{4} \right) - \sqrt{2}(N + 2l + 8) - 2\lambda_2 = 0, \quad (3.89)$$

$$\frac{1}{2} \left( \lambda_3 - \frac{\lambda_4^2}{4} \right)^2 - \frac{\lambda_4}{\sqrt{2}}(N + 2l + 6) - 2\lambda_1 = 4\sqrt{2}z_1. \quad (3.90)$$

The energy eigenvalue  $E$  is given by

$$E = \frac{1}{2\sqrt{2}} \left( \lambda_3 - \frac{\lambda_4^2}{4} \right) (N + 2l + 4) + \sqrt{2}\lambda_4 z_1 + 2\sqrt{2}z_1^2, \quad (3.91)$$

and the radial wave function is

$$R(r) = r^l (r^2 - z_1) \exp \left[ -\frac{1}{2\sqrt{2}} \left( \lambda_3 - \frac{\lambda_4^2}{4} \right) r^2 - \frac{\lambda_4}{4\sqrt{2}} r^4 - \frac{1}{3\sqrt{2}} r^6 \right] \quad (3.92)$$

with  $z_1$  given by (3.87). This gives the first excited state of the system.

## 4 Concluding remarks

The main results of this paper are theorem 1.1, corollary 1.3 and their applications to the exact solutions of the physical systems (examined in section 3).

As spelled out in theorem 1.1, we have found all polynomials  $Z(z)$  such that the ODE (1.1) has polynomial solutions  $S(z)$  of degree  $n$  with distinct roots  $z_1, z_2, \dots, z_n$ . We have also found a set of  $n$  algebraic equations which determine the roots  $z_i$  and thus the corresponding polynomial solutions  $S(z)$ . If the coefficients of polynomials  $X(z)$  and  $Y(z)$  in (1.1) are algebraically dependent, i.e satisfy the relations in corollary 1.3, then ODE (1.1) allows an  $sl(2)$  algebraization. We have also found all the polynomials  $Z(z)$  and degree  $n$  polynomial solutions  $S(z)$  under these conditions.

Having as many as 12 parameters, the ODE (1.1) is very general and contains, as special cases, most known second order differential equations which occur in physical, chemical and engineering applications in the literature (e.g. the relatively simple Heun equation and its various confluent equations). Thus our general results (theorem 1.1 and its corollaries) provide an unified derivation of exact, closed form solutions for all such systems. As applications to theorem 1.1, we have examined five physical systems in section 3, which are described by the ODEs corresponding to (1.2)-(1.6), respectively. We have shown that these systems are quasi-exactly solvable, i.e. the corresponding differential equations have polynomial solutions, if their parameters satisfy certain constraints (special cases of (1.8)-(1.10)). The quasi-exact solvability has enabled us to use theorem 1.1 to obtain the closed form expressions for the eigenvalues and eigenfunctions of these systems.

Our results (theorem 1.1 and corollary 1.3) can be extended to second order ODE of the form (1.1) with  $\deg X(z) \leq l$ ,  $\deg Y(z) \leq l - 1$  and  $\deg Z(z) \leq l - 2$  for  $l \geq 5$  as

well as to higher order ODEs. Research on this as well as on applications of theorem 1.1 and corollary 1.3 in various areas of science is in progress, and results will be reported elsewhere.

**Acknowledgments:** I would like to thank Ryu Sasaki for a very careful reading of the manuscript and many critical comments, and Clare Dunning for a careful reading of the manuscript and helpful suggestions. I also thank Tony Bracken, Günter von Gehlen, Mark Gould, and Jon Links for comments and suggestions. This work was supported by the Australian Research Council.

## 5 Appendix

For completeness and convenience of applications, in this appendix we write down the explicit formulas obtained from applying the theorem 1.1 to the special cases (1.2)-(1.6).

**Corollary 5.1** *The coefficients  $c_1, c_0$  of  $Z(z)$  such that the Heun equation (1.2) has polynomial solution (1.7) are given by*

$$c_1 = -n \left[ n - 1 + \sum_{s=1}^3 \alpha_s \right], \quad (5.1)$$

$$c_0 = - \left[ 2(n-1) + \sum_{s=1}^3 \alpha_s \right] \sum_{i=1}^n z_i + n(n-1) \sum_{s=1}^3 d_s + n[\alpha_1(d_2 + d_3) + \alpha_2(d_1 + d_3) + \alpha_3(d_1 + d_2)], \quad (5.2)$$

where the roots  $z_1, z_2, \dots, z_n$  are determined by the Bethe ansatz equations,

$$\sum_{j \neq i}^n \frac{2}{z_i - z_j} + \sum_{s=1}^3 \frac{\alpha_s}{z_i - d_s} = 0, \quad i = 1, 2, \dots, n. \quad (5.3)$$

Write  $c_1 = \alpha\beta$ . Then (5.1) is nothing but the so-called Fuchsian relation,  $\alpha + \beta + 1 = \sum_{s=1}^3 \alpha_s$ , where  $\alpha = -n$  and  $\beta = \sum_{s=1}^3 \alpha_s + n - 1$ .

**Corollary 5.2** *The coefficients  $c_2, c_1, c_0$  of  $Z(z)$  such that the generalized Heun equation (1.3) has polynomial solution (1.7) are*

$$c_2 = -n \left( \sum_{s=1}^4 \mu_s + n - 1 \right), \quad (5.4)$$

$$c_1 = - \left( \sum_{s=1}^4 \mu_s + 2(n-1) \right) \sum_{i=1}^n z_i + n \left[ (n-1) \sum_{s=1}^4 e_s + P \right], \quad (5.5)$$

$$c_0 = - \left( \sum_{s=1}^4 \mu_s + 2(n-1) \right) \sum_{i=1}^n z_i^2 + 2 \sum_{i < j}^n z_i z_j$$

$$- \left[ 2(n-1) \sum_{s=1}^4 e_s + P \right] \sum_{i=1}^n z_i + n(n-1) \sum_{s < t}^4 e_s e_t + Qn, \quad (5.6)$$

where

$$\begin{aligned} P &= \mu_1(e_2 + e_3 + e_4) + \mu_2(e_1 + e_3 + e_4) + \mu_3(e_1 + e_2 + e_4) + \mu_4(e_1 + e_2 + e_3), \\ Q &= \mu_1(e_2 e_3 + e_2 e_4 + e_3 e_4) + \mu_2(e_1 e_3 + e_1 e_4 + e_3 e_4) \\ &\quad + \mu_3(e_1 e_2 + e_1 e_4 + e_2 e_4) + \mu_4(e_1 e_2 + e_1 e_3 + e_2 e_3), \end{aligned}$$

and the roots  $z_1, z_2, \dots, z_n$  are determined by the Bethe ansatz equations,

$$\sum_{j \neq i}^n \frac{2}{z_i - z_j} + \sum_{s=1}^4 \frac{\mu_s}{z_i - e_s} = 0, \quad i = 1, 2, \dots, n. \quad (5.7)$$

**Corollary 5.3** *The coefficients  $c_2, c_1, c_0$  of  $Z(z)$  such that the differential equation (1.4) has polynomials solution (1.7) are given by*

$$c_2 = -n\nu, \quad (5.8)$$

$$c_1 = -\nu \sum_{s=1}^n z_i - n \left[ (n-1) + \sum_{s=1}^3 \nu_s - \nu \sum_{s=1}^3 f_s \right], \quad (5.9)$$

$$\begin{aligned} c_0 &= -\nu \sum_{s=1}^n z_i^2 - \left[ 2(n-1) + \sum_{s=1}^3 \nu_s - \nu \sum_{s=1}^3 f_s \right] \sum_{i=1}^n z_i + n(n-1) \sum_{s=1}^3 f_s \\ &\quad + [\nu(f_1 f_2 + f_1 f_3 + f_2 f_3) - \nu_1(f_2 + f_3) - \nu_2(f_1 + f_3) - \nu_3(f_1 + f_2)] n, \end{aligned} \quad (5.10)$$

where the roots  $z_1, z_2, \dots, z_n$  satisfy the Bethe ansatz equations,

$$\sum_{j \neq i}^n \frac{2}{z_i - z_j} + \sum_{s=1}^3 \frac{\nu_s}{z_i - f_s} + \nu = 0, \quad i = 1, 2, \dots, n. \quad (5.11)$$

**Corollary 5.4** *The coefficients  $c_2, c_1, c_0$  of  $Z(z)$  such that the differential equation (1.5) has polynomials solution (1.7) are*

$$c_2 = -n\sigma, \quad (5.12)$$

$$c_1 = -\sigma \sum_{s=1}^n z_i - n [\kappa - \sigma(g_1 + g_2)], \quad (5.13)$$

$$\begin{aligned} c_0 &= -\sigma \sum_{s=1}^n z_i^2 - [\kappa - \sigma(g_1 + g_2)] \sum_{i=1}^n z_i \\ &\quad - n(n-1) - n [\sigma_1 + \sigma_2 + \sigma g_1 g_2 - \kappa(g_1 + g_2)], \end{aligned} \quad (5.14)$$

where the roots  $z_1, z_2, \dots, z_n$  obey the Bethe ansatz equations,

$$\sum_{j \neq i}^n \frac{2}{z_i - z_j} + \frac{\sigma_1}{z_i - g_1} + \frac{\sigma_2}{z_i - g_2} + \sigma z_i + \kappa = 0, \quad i = 1, 2, \dots, n. \quad (5.15)$$

**Corollary 5.5** *The coefficients  $c_2, c_1, c_0$  of  $Z(z)$  such that the differential equation (1.6) has polynomials solution (1.7) are given by*

$$c_2 = -n\lambda, \quad (5.16)$$

$$c_1 = -\lambda \sum_{s=1}^n z_s - n(\gamma - \lambda h), \quad (5.17)$$

$$c_0 = -\lambda \sum_{s=1}^n z_s^2 - (\gamma - \lambda h) \sum_{i=1}^n z_i - n(\delta - \gamma h), \quad (5.18)$$

where the roots  $z_1, z_2, \dots, z_n$  are determined by the Bethe ansatz equations,

$$\sum_{j \neq i}^n \frac{2}{z_i - z_j} + \frac{\eta}{z_i - h} + \lambda z_i^2 + \gamma z_i + \delta = 0, \quad i = 1, 2, \dots, n. \quad (5.19)$$

## References

- [1] R. Schäfke and D. Schmidt, The connection problem for general linear ordinary differential equations at two regular singular points with applications in the theory of special functions, *SIAM J. Math. Anal.* **11** (1980), 848-862.
- [2] D. Gómez-Ullate, N. Kamran and R. Milson, An extended class of orthogonal polynomials defined by a Sturm-Liouville problem, *J. Math. Anal. Appl.* **359** (2009), 352-367; An extension of Bochner's problem: exceptional invariant subspaces, *J. Approx. Theory* **162** (2010), 987-1006.
- [3] C-L. Ho, S. Odake and R. Sasaki, Properties of the exceptional  $(X_\ell)$  Laguerre and Jacobi polynomials, arXiv:0912.5477v3 [math-ph], and references therein.
- [4] J. Borcea and B. Shapiro, Root asymptotics of spectral polynomials for the Lamé operator, arXiv:math/0701883v2 [math.CA].
- [5] B. Shapiro and M. Tater, On spectral polynomials of the Heun equations. I., arXiv:0812.2321v1 [math-ph].  
B. Shapiro, K. Takemura and M. Tater, On spectral polynomials of the Heun equations. II., arXiv:0904.0650v1 [math-ph].
- [6] I. Scherbak and A. Varchenko, Critical points of functions,  $sl_2$  representations, and Fuchsian differential equations with only univalued solutions, arXiv:math/0112269v4 [math.QA].
- [7] E. Mukhin, V. Tarasov and A. Varchenko, Higher Lamé equations and critical points of master functions, arXiv:math/0601703v2 [math.CA].

- [8] A. Turbiner, CRC Handbook of Lie Group Analysis of Differential Equations, Vol. **3**, Chap. 12, ed. N. H. Ibragimov, CRC Press, Boca Raton, FL, 1996.
- [9] A. González-López, N. Kamran and P. Olver, Normalizability of one-dimensional quasi-exactly solvable Schrödinger operators, *Commun. Math. Phys.* **153** (1993), 117-146.
- [10] C.M. Bender and G.V. Dunne, Quasi-exactly solvable systems and orthogonal polynomials, *J. Math. Phys.* **37** (1996), 6-11.
- [11] A. Krajewska, A. Ushveridze and Z. Walczak, Bender-Dunne orthogonal polynomials and quasi-exact solvability, arXiv:hep-th/9601088v1.
- [12] A.G. Ushveridze, Quasi-exactly solvable models in quantum mechanics, Institute of Physics Publishing, Bristol, 1994.
- [13] E. Heine, Handbuch der Kugelfunctionen, Vol. **1**, pp. 472-479, D. Reimer Verlag, Berlin, 1878.
- [14] T. Stieltjes, Sur certains polynômes qui vérifient une equation différentielle linéaire du second ordre et sur la theorie des fonctions de Lamé, *Acta Math.* **8** (1885), 321-326.
- [15] G. Szego, Orthogonal polynomials, American Mathematical Society, 1939.
- [16] P.B. Wiegmann and A.V. Zabrodin, Bethe ansatz for the Bloch electron in magnetic field, *Phys. Rev. Lett.* **72** (1994), 1890-1893; Algebraization of difference eigenvalue equations related to  $U_q(sl_2)$ , *Nucl. Phys. B* **451** (1995), 699-724.
- [17] R. Sasaki, W.-L. Yang and Y.-Z. Zhang, Bethe ansatz solutions to quasi-exactly solvable difference equations, *SIGMA* **5** (2009), 104 (16 pages).
- [18] Y.-H. Lee, W.-L. Yang and Y.-Z. Zhang, Polynomial algebras and exact solutions of general quantum non-linear optical models I: Two-mode boson systems, *J. Phys. A: Math. Theor.* **43** (2010), 185204 (17 pages); Polynomial algebras and exact solutions of general quantum non-linear optical models II: Multi-mode boson systems, *J. Phys. A: Math. Theor.* **43** (2010), 375211 (12 pages).
- [19] Y.-H. Lee, J.R. Links and Y.-Z. Zhang, Exact solutions for a family of spin-boson systems, *Nonlinearity* **24** (2011), 1975-1786.
- [20] P-F. Loos and P.M.W. Gill, Two electrons on hypersphere: a quasi-exactly solvable model, *Phys. Rev. Lett.* **103** (2009), 123008 (4 pages); Excited states of spherium, *Mol. Phys.* **108** (2010), 2527-2532.



- [21] N.H. Christ and T.D. Lee, Quantum expansion of soliton solutions, *Phys. Rev. D* **12** (1975), 1606-1627.
- [22] D.P. Jatkar, C.N. Kumar and A. Khare, A quasi-exactly solvable problem without  $sl(2)$  symmetry, *Phys. Lett. A* **142** (1989), 200-202, and references therein.
- [23] S.S. Gubser, Phase transitions near black hole horizons, *Class. Quant. Grav.* **22** (2005), 5121-5143.
- [24] D. Batic, H. Schmid and M. Winklmeier, The generalized Heun equation in QFT in curved space-time, *J. Phys. A: Math. Gen.* **39** (2006), 12559-12564.
- [25] C.-M. Chiang and C.-L. Ho, Planar Dirac electron in Coulomb and magnetic fields, *J. Math. Phys.* **43** (2002), 43-51.
- [26] F. Pan, J.R. Klauder and J.P. Draayer, Quasi-exactly solvable case of an  $N$ -dimensional symmetric decatic anharmonic oscillator, *Phys. Lett. A* **262** (1999), 131-136.